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Quantum Structures Do Not Exist in Reality

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It is argued that quantum logic and quantum probability theory are fascinating mathematical theories but without any relevance to our real world.

1. INTRODUCTION

If *quantum probability theory* takes its name seriously, then it must include at least two components: $(\mathscr{A}, p)$. $\mathscr{A}$ is an algebraic structure, such that the following representation is provided:

$$\text{ELEMENTS} \Leftrightarrow \begin{array}{l}\text{EVENTS}\\ \text{(e.g., beeps of a detector,}\\ \text{pointer positions, etc.)}\end{array}$$

$$\text{OPERATIONS} \Leftrightarrow \begin{cases}\text{JOINT OCCURRENCE}\\ \text{ALTERNATIVE OCCURRENCE}\\ \text{NONOCCURRENCE}\end{cases}$$

p is a function, $p: \mathscr{A} \rightarrow [0, 1]$, satisfying some elementary conditions, and the value of which must be close to the observed relative frequencies.

The story of quantum probability and quantum logic begins with von Neumann's (Birkoff and Neumann 1936) recognition that quantum mechanics *can be regarded* as a kind of probability theory defined over the subspace lattice $L(H)$ of a Hilbert space H. This recognition was confirmed by the Gleason theorem:

Definition 1. A nonnegative real function μ on $L(H)$ is called a probability measure if $\mu(H) = 1$ and if, whenever $E_1, E_2, \ldots$ are pairwise orthogonal subspaces and $E = \vee_{i=1}^{\infty} E_i$, then $\mu(E) = \Sigma_{i=1}^{\infty} \mu(E_i)$.

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Theorem 1 (Gleason, 1957). If H is a real or complex Hilbert space of dimension greater than 2, and μ is a probability measure on $L(H)$, then there exists a density operator W on H such that $(\forall E \in L(H))\ [\mu(E) = \mathrm{tr}(WE)]$.[2]

Formally, the closed linear union and the intersection of subspaces play the role of disjunction and conjunction of events in this probability theory. That is, for example, a conjunction $A \wedge B$, represented by the intersection of the corresponding subspaces $A \cap B$, corresponds to an event which is nothing else but the *joint occurrence* of events A and B.

This is, however, only a mathematical abstraction and, as we will see soon, the above interpretation is untenable.

2. NONSENSICAL "PROBABILITIES"

Consider the example shown in Fig. 1. Let E_1 and E_2 be one-dimensional subspaces of a two-dimensional Hilbert space H^2. Ψ is the state vector of the system. In this state, the corresponding probabilities are the following:

$$p(E_1) = \langle \Psi, E_1\Psi \rangle = 1$$

$$p(E_2) = \langle \Psi, E_2\Psi \rangle = 0.6\ (>0!)$$

$$p(e_1 \wedge E_2) = \langle \Psi, (E_1 \wedge E_2\Psi) \rangle = 0$$

The strange meaning of this result is obvious: if E_1 happens with certainty, how can E_2 occur without E_1? Consequently.

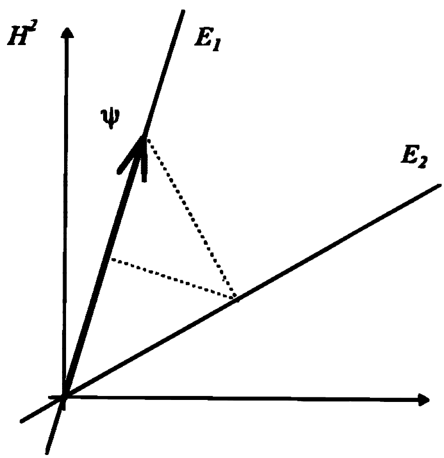

Fig. 1. A simple two-dimensional example for nonsensical probabilities.

[2] The subspaces, the corresponding projectors, and the corresponding events are denoted by the same letter.

1. $L(H)$ can hardly play the role of an "algebra of events" for a probability theory.
2. The number $\mathrm{tr}(WE)$ cannot be interpreted as the "relative frequency" of an event.

3. DO WE REALLY NEED A NONCLASSICAL PROBABILITY THEORY?

Until we restrict ourselves to one set of commuting observables, quantum probabilities can be represented in one Kolmogorovian probability space. However, we can see in many often-quoted examples, such as the double-slit experiment, the EPR experiment, etc., that quantum mechanics produces different Kolmogorovian probability measures belonging to different incompatible conditions.[3] The alleged impossibility to put these classical probability measures together into *one common* Kohnogorovian probability model is behind the cry for quantum probability theory and quantum logic. The principal point of my claim is that we *can* join these probability measures, if we do it in a correct way!

Before seeing how can we do that, let us consider how this procedure goes in the classical theory of probability.

3.1. How to Unify Classical Probability Models?

Let me take a simple example. We toss a coin which has a little magnetic momentum (Fig. 2). If the magnetic field is off, the probabilities are

$$p_{\mathrm{off}}(\mathrm{H}) = 0.5$$

$$p_{\mathrm{off}}(\mathrm{T}) = 0.5$$

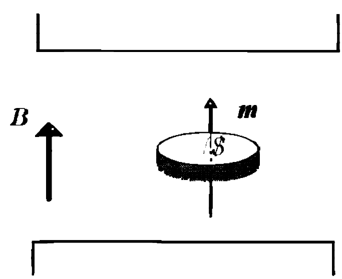

Fig. 2. The probabilities of Heads (H) and Tails (T) are different if the magnetic field is on.

[3]For a detailed analysis of these examples see Szabó (1995a, b, 1996).

If the magnetic field is on, the probabilities are different:

$$p_{\text{on}}(\text{H}) = 0.2$$

$$p_{\text{on}}(\text{T}) = 0.8$$

The event algebra $\mathscr{A}$ is shown in Fig. 3. Probability models $(\mathscr{A}, p_{\text{off}})$ and $(\mathscr{A}, p_{\text{on}})$ are, separately, Kolmogorovian. For example, they satisfy some elementary Bell-type inequalities (Pitowsky, 1989):

$$p_{\text{off}}(\text{H}) + p_{\text{off}}(\text{T}) - p_{\text{off}}(\text{H} \wedge \text{T}) \leq 1 \tag{1}$$

and separately,

$$p_{\text{on}}(H) + p_{\text{on}}(T) - p_{\text{on}}(H \wedge T) \leq 1 \tag{2}$$

Now, if we make the same mistake as we do so often in quantum mechanics, and put these probabilities, belonging to different conditions, together into one formula prescribed for a Kolmogorovian probability theory, we find the same kind of "violation of the rules of classical probability theory":

$$p_{\text{off}}(\text{H}) + p_{\text{on}}(\text{T})) - p_{\text{off}}(H \wedge T) = 0.5 + 0.8 > 1 \tag{3}$$

or

$$p_{\text{on}}(\text{H}) + p_{\text{off}}(T) = 0.2 + 0.5 \neq 1 = p_{\text{off}}(1) = p_{\text{off}}(\text{H} \vee T) \tag{4}$$

Consider now how to join probability models $(\mathscr{A}, p_{\text{off}})$ and $(\mathscr{A}, p_{\text{on}})$. In the classical probability theory we can join probabilities belonging to separate conditions only by enlarging the event algebra in such a way that it contains not only the original events, but the "conditioning events" too (Fig. 4).[4] Of course, we can do that only if we know the probabilities of the conditioning

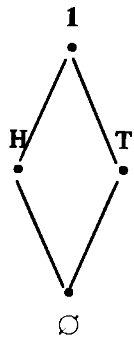

Fig. 3. Algebra of events $\mathscr{A}$.

[4] I am grateful to Miltos Zissis for his warning that Fig. 4 was incorrect in a previous version of this paper.

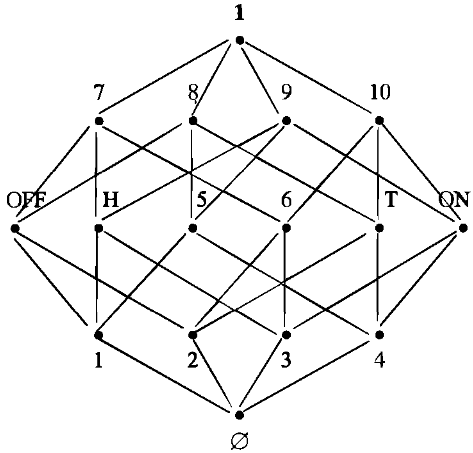

Fig. 4. The unified algebra of events $\mathscr{A}'$.

events. In the example in question assume that $p\,(\mathrm{OFF}) = 0.5$ and $p\,(\mathrm{ON}) = 0.5$. So, the unified probability model is $(\mathscr{A}', p)$, where

$$
\begin{aligned}
p\,(1) = p\,(2) = p\,(9) = p\,(10) &= 0.25 \\
p\,(3) = p\,(8) &= 0.1 \\
p\,(4) = p\,(7) &= 0.4 \\
p\,(\mathrm{OFF}) = p\,(\mathrm{ON}) &= 0.5 \\
p\,(\mathrm{H}) = p\,(6) &= 0.35 \\
p\,(\mathrm{T}) = p\,(5) &= 0.65
\end{aligned}
\tag{5}
$$

The original probabilities are represented as conditional probabilities (defined by the Bayes law):

$$
\begin{aligned}
p_{\mathrm{on}}(\mathrm{H}) &= \frac{p\,(\mathrm{H} \wedge \mathrm{ON})}{p\,(\mathrm{ON})} = \frac{p\,(3)}{p\,(\mathrm{ON})} = \frac{0.1}{0.5} = 0.2 \\
p_{\mathrm{on}}(\mathrm{T}) &= \frac{p\,(\mathrm{T} \wedge \mathrm{ON})}{p\,(\mathrm{ON})} = \frac{p\,(4)}{p\,(\mathrm{ON})} = \frac{0.4}{0.5} = 0.8 \\
p_{\mathrm{off}}(\mathrm{H}) &= \frac{p\,(\mathrm{H} \wedge \mathrm{OFF})}{p\,(\mathrm{OFF})} = \frac{p\,(1)}{p\,(\mathrm{OFF})} = \frac{0.25}{0.5} = 0.5 \\
p_{\mathrm{off}}(\mathrm{T}) &= \frac{p\,(\mathrm{T} \wedge \mathrm{OFF})}{p\,(\mathrm{OFF})} = \frac{p\,(2)}{p\,(\mathrm{OFF})} = \frac{0.25}{0.5} = 0.5
\end{aligned}
\tag{6}
$$

3.2. How to Join Probability Distributions in Quantum Theory?

Consider a quantum system described in Hilbert space H. The state of the system is represented by density operator W. Assume that there are N different measurements $m_1, m_2, \ldots, M_N$ one can carry out on the system. The corresponding observable-operators are denoted by $\hat{M}_1, \hat{M}_2, \ldots, \hat{M}_N$. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_N$ be the spectra of these operators. Introduce the following notation: each set of measurements $\{m_{\iota_1}, \ldots m_{i_s}\{$ will be identified with a vector $\eta \in \{0, 1\}^n$, such that

$$\eta_i = \begin{cases} 1 & \text{if} \quad m_i \in \{m_{1_1}, \ldots m_{i_s}\} \\ 0 & \text{if} \quad m_i \notin \{m_{2_1}, \ldots m_{i_s}\} \end{cases}$$

In this way the conditioning events can be represented in $2^{\{0,1\}^N}$. For instance, event "measurement m_i is performed" is represented by $\{\eta | \eta_i = 1\} \subset \{0, 1\}^N$, event "measurement m_ι and measurement m_j are performed" corresponds to

$$\{\eta | \eta_i = 1\} \cap \{\eta | \eta_j = 1\}$$

etc.

Some of these measurements can be incompatible, in the sense that they cannot be simultaneously carried out. Assume that, according to the quantum theory, observables belonging to one set of compatible measurements commute. For each set $\{m_{i_1}, m_{i_2}, \ldots m_{i_s}\} = \{m_i\}_{\eta_i=1}$ of *compatible* measurements the quantum state W determines a Kolmogorovian probability measure over the corresponding Borel sets, $(B(\times_{\eta_i=1} \mathcal{M}_i), \mu_\eta)$, where

$$\mu_\eta: \quad (A_i)_{\eta_i=1} \in B\left(\underset{\eta_i=1}{\times} \mathcal{M}_i \right) \mapsto \mathrm{tr}\left(W \prod_{\eta_i=1} A_i \right) \tag{7}$$

Now, how can we join these classical probability spaces into one common classical probability model? The method is known from the classical theory of probability. Quantum mechanics has nothing special from this point of view! That is, we need to enlarge the event algebra by the conditioning events and to define the joint probability measure over this larger algebra of events. In order to do that, we need to know the probabilities of conditioning events. The values of these probabilities are the matter of empirical facts, although the following assumption seems to be quite plausible:

Stipulation. There is a classical probability measure $\tilde{p}$ on $2^{\{0,1\}^N}$ such that if $\tilde{p}(\{\eta\}) \neq 0$, then the corresponding set of operators $\{M_i\}_{\eta_i=1}$ is commuting.

Thus, my assertion is that classical probabilities (7) can be joined into one Kolmogorovian probability model:

Theorem 2. There exists a Kolmogorovian probability space ($\mathcal{M}$, $B(\mathcal{M})$, p) such that each conditioning event $E \in 2^{\{0,1\}^N}$ and each outcome event $(A_i)_{\eta_i=1}$ can be represented by an element of $B(\mathcal{M})$, denoted by X_E and $X_{(A_i)_{\eta_i=1}}$, respectively, and

$$\bar{p}(E) = p(X_E), \qquad \forall E \in 2^{\{0,1\}^N}$$

$$\mu_\eta((A_i)_{\eta_i=1}) = \mathrm{tr}\left(W \prod_{\eta_i=1} A_i\right)$$

$$= \frac{p(X_{(A_i)_{\eta_i=1}} \cap X_{\{\eta\}})}{p(X_{\{\eta\}})}, \qquad \forall \eta \in \{0, 1\}^N, \quad \forall (A_i)_{\eta_i=1} \in \times_{\eta_i=1} \mathcal{M}_i$$

For the proof of this statement and for further details see Szabó (1996).

4. CONCLUSIONS

Like it or not, quantum mechanics is connected with the empirical facts about the world, to which it is supposed to be applied, through relative frequencies. But those "probabilities" that are presented by the quantum probability theory can hardly be interpreted as relative frequencies of events. And whether we like it or not, quantum logic is nothing else but an algebraic structure isomorphic with the algebra of events underlying the quantum probability theory. So, if quantum probability theory has nothing to do with reality, then quantum logic is meaningless, too. Moreover, we have seen that the main motivation of quantum probability theory is groundless: it is based on the complaint that probability measures belonging to different set of conditions cannot be unified into one common Kolmogorovian probability model. But the manner in which this unification is imagined is mistaken. In this way one could run into contradictions even in the classical probability theory. Finally, it can be proved that such a unification of "incompatible" probability measures, if it is understood correctly, is entirely possible in quantum mechanics.

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